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Stability of convection in containers of arbitrary shape

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When a container of fluid of arbitrary shape is heated from below and the temperature gradient exceeds a critical value ($\mathscr{R}_c^2$) the conduction solution with no motion becomes unstable and is replaced by convection. The convection may have two forms: one with 'upflow' at the centre of the container and one with 'downflow' there. Here we study the stability of the two forms of convection. Both forms are here shown to be stable to infinitesimal disturbances. When the viscosity varies with the temperature or the conduction profile is not linear, etc., the steady convection can be driven with finite amplitudes $|\epsilon|$ at subcritical values of the temperature contrast ($\mathscr{R}^2 < \mathscr{R}_c^2$). This subcritical convection is stable when the convection is strong ($|\epsilon| > |\epsilon_*| > 0$) but is unstable when the convection is feeble ($|\epsilon| < |\epsilon_*|$). Hence, when $|\epsilon| > |\epsilon_*|$ and $\mathscr{R}^2 < \mathscr{R}_c^2$ either 'upflow' or 'downflow', but not both, is stable. When $\mathscr{R}^2 > \mathscr{R}_c^2$, however, both the 'upflow' and the 'downflow' can be stable. This contrasts with the corresponding situation which is known to hold when the container is an unbounded layer. In the layer there is only one stable form of convection. The difference between the bounded domain with two forms of convection and the layer with just one stable form is traced to the mathematical property of simplicity of $\mathscr{R}_c^2$ when viewed as an eigenvalue of the linear stability problem for the conduction solution. It is argued that $\mathscr{R}_c^2$ is a simple eigenvalue in most domains, but in the layer $\mathscr{R}_c^2$ can have infinite multiplicity. The explanation of the transition from the bounded domain to the unbounded layer is sought (1) in the chaotic conditions which frequently prevail at the edges of a 'bounded' layer and (2) in the fact that in the layer of large horizontal extent, the higher eigenvalues crowd $\mathscr{R}_c^2$. In the course of the explanation, a new exact solution of the linear Bénard problem in a cylinder with a rigid side wall and a stress-free top and bottom is derived.

1. Introduction

It is well known that the motionless state of pure heat conduction is stable to small disturbances when $\mathscr{R} < \mathscr{R}_c$, where $\mathscr{R}_c$ is a critical value of linear stability theory. There are many problems which allow stable convection as well as stable conduction to exist when $\mathscr{R} < \mathscr{R}_c$. In these problems the appearance of conduction or convection at $\mathscr{R} < \mathscr{R}_c$ cannot be decided by linear theory. (We need to know the 'domains of attraction' of the linearly stable solutions.) A class of problems of this type arises in the study of generalized conduction solutions of the generalized *Oberbeck–Boussinesq* equations.† A conduction solution is a motionless solution of the governing equations. All such solutions necessarily have parallel gravity and temperature gradient fields. The conduction solution is called 'generalized' if the thermal diffusivity is temperature dependent, if the external conditions vary with time or if there are heat sources in the fluid. The generalizations of the O–B equations to be kept in mind allow thermal properties of the fluid and the derivative of the density with respect to temperature to vary with the temperature. But we shall not lose generality if, for definiteness, we consider a fluid with a temperature-sensitive viscosity in a container heated from below and by heat sources.

Let ξ be a heat source parameter (equation (2.3)), let ζ be a viscosity parameter (equation (2.2)) and let ϵ be the amplitude of the steady convection (equation (2.6*d*)). To drive convection with norm $|\epsilon|$, one needs to impose a definite (dimensionless) temperature difference $\mathscr{R}(\epsilon, \xi, \zeta)$ across the container, but more than one convective solution can often be found for a given value of $\mathscr{R}$ (see figure 1). The convective solutions with small $|\epsilon|$ branch off the conduction solution at the value $\mathscr{R}_c = \mathscr{R}(0, \xi, \zeta)$ and are sometimes called 'branching' solutions. For $\mathscr{R} < \mathscr{R}_c$, the conduction solution is stable, and when $\mathscr{R} > \mathscr{R}_c$, the conduction solution loses stability.

The number of feeble convective solutions (ϵ near 0) is very much related to the number of linearly independent eigenfunctions which belong to the eigenvalue

† By O–B equations I have in mind that generalization of the incompressible Navier–Stokes equations which is generally attributed to Boussinesq (1903). It is known, through the work of Mihaljan (1962), that the approximations used by Boussinesq were actually of earlier origin and were used by Oberbeck (1891) in meteorological studies of the Hadley régime. But Oberbeck's first use of these equations, in 1879, is more substantial than the later (1891) application, and the equations which he sets out in the earlier study are just exactly the ones generally attributed to Boussinesq. In some respects, Oberbeck's (1879) treatment of these equations is superior to Boussinesq's; for example, Boussinesq obtains the simplified equations as a consequence of a list of assertions, but in Oberbeck's work the equations arise as the lowest-order terms of a power series development in the expansion coefficient α. And included in Oberbeck's fundamental paper is an application of both the convection analysis and the series ordering by which they are derived to the problem of convection induced by differential heating of stationary concentric spheres.

It cannot be said that the Oberbeck study went unnoticed, for in 1881, Lorenz published a celebrated study of free convection along a heated flat plate in which he used Oberbeck's equations to derive a formula relating, for the first time, the Nusselt number to given flow data. Jakob (1949, p. 443) calls this formula "...a triumph of classical theory, having revealed for the first time the complex nature of the coefficient of heat transfer by free convection...in a form which has been proved valid with good approximation through more than half a century". The 'Rayleigh' number also appears for the first time in the Lorenz study (it is called α there and $\mathscr{R}^2$ here).

The attribution of the equations to Boussinesq is probably due to Rayleigh (1916), who may have been unaware of the earlier papers, though it cannot be said that these earlier papers are in any sense obscure. Such was the prestige of Rayleigh, especially in England, that this attribution stuck, despite the fact that the engineering heat transfer literature makes use of the 'Boussinesq' equations as if they were created at $t = 0$, without reference to Boussinesq but with a properly humble deference to the important results of Lorenz.

$\mathscr{R}_c$. The multiplicity of $\mathscr{R}_c$ is the number of independent eigensolutions. When there is only one eigenfunction, the eigenvalue (of multiplicity one) is called simple. In the fluid layer the multiplicity of $\mathscr{R}_c$ can be very large, even infinite.† By requiring symmetry properties of solutions (e.g. Yudovich 1966), one can make $\mathscr{R}_c$ simple. But even when $\mathscr{R}_c$ is simple, there are still *two* convection solutions. On one solution the fluid rises at the 'centre' and on the other solution the fluid sinks at the 'centre'.

Both the rising and falling solution can be constructed formally as series in ϵ (Malkus & Veronis 1954) and also in ξ and ζ (Busse 1962). The formal series converge and solve the problem (Fife & Joseph 1969, Fife 1970). We shall express the steady convection as Taylor series

$$\mathbf{u} = \epsilon(\mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \ldots), \tag{1.1a}$$

$$\theta = \epsilon(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \ldots), \tag{1.1b}$$

and

$$\mathscr{R} = \mathscr{R}_c + \epsilon\mathscr{R}_1 + \epsilon^2\mathscr{R}_2 + \ldots, \tag{1.1c}$$

etc. The linear stability problem for the steady convection will also be expressed in series (4.1, 4.5*b*). Here the coefficients are Taylor coefficients, e.g.

$$\theta_2 = (1/2!)\,\partial^2\theta/\partial\epsilon^2|_{\epsilon=0},$$

and $\mathbf{u}$ and θ are the velocity and temperature of a disturbance of the conduction solution. The functions $(\mathbf{u}_0, \theta_0)$ have $\mathscr{R}_c$ as their eigenvalue and since $\mathscr{R}_c$ is to be assumed simple, these functions are fixed to within an arbitrary multiplicative constant. The magnitude of the multiplicative constant is fixed by condition (2.6*d*), which defines ϵ^2. The sign of the constant is not fixed by (2.6*d*) and $(\mathbf{u}_0, \theta_0)$ and $(-\mathbf{u}_0, -\theta_0)$ are both eigenfunctions of $\mathscr{R}_c$. If $(\epsilon\mathbf{u}_0, \epsilon\theta_0)$ gives a rising motion at a point, $(-\epsilon\mathbf{u}_0, -\epsilon\theta_0)$ gives a sinking motion. It is convenient to think of the functions $(\mathbf{u}_0, \theta_0)$ as given. Then, the two motions are associated with the sign of ϵ. For example, we could choose the sign of $(\mathbf{u}_0, \theta_0)$ so that $\epsilon > 0$ corresponds to rising motion and $\epsilon < 0$ to sinking motion at the centre.

When $\xi^2 + \zeta^2 > 0$ subcritical convection ($\mathscr{R} < \mathscr{R}_c$) is possible. Suppose, for definiteness, that this subcritical motion has $\epsilon > 0$ and is associated with the rising motion. As ϵ is increased $\mathscr{R}(\epsilon)$ decreases to a minimum, say $\mathscr{R}(\epsilon_*)$, and then increases again. For $\epsilon < 0$ the convection is supercritical ($\mathscr{R}(\epsilon) > \mathscr{R}_c$). Therefore the convection can be supercritical when $\epsilon < 0$ or $\epsilon > \epsilon_* > 0$. Rising and falling motion can occur when $\mathscr{R} > \mathscr{R}_c$ but only rising motion ($\epsilon > 0$) can occur when $\mathscr{R} < \mathscr{R}_c$ (see figure 1).

We will consider the stability of these different forms of convection. In the fluid layer the only stable form of convection has $\epsilon > \epsilon_*$. Here $\mathscr{R}(\epsilon)$ increases with

† As is well known (Pellew & Southwell 1940) the linear stability problem for O–B convection allows a 'so-called' plan form separation of variables of the form $f(z)g(x, y)$ where g is an eigenfunction of the Laplacian in the plane with wave-number a^2, i.e. $\Delta_2 g + a^2 g = 0$. There are infinitely many such solutions even when a^2 is fixed. If a_c^2 is the wave-number associated with the eigenvalue $\mathscr{R}_c$, all the eigensolutions are of the form $f(z)g_c(x, y)$, $\Delta_2 g_c + a_c^2 g_c = 0$ and these are infinite in number.

ϵ. Moreover, the stable solution has a hexagonal plan form.† It follows that stability determines the sign of the motion in fluid layers.‡

In contrast, in bounded regions both the rising and falling motion are not only possible – they are both stable to small disturbances.

Since in the layer only one of the two branching solutions is stable, the sign of ϵ for the stable solution, that is, the sign of the motion, is uniquely determined. This stability consideration is important in, say, the generally accepted view that the direction of flow in cellular convection is determined by the variation of the viscosity with the temperature, and is such that the motion at the centre of the cell is in the direction of increasing kinematic viscosity (the subcritical branch). This view was first advanced by Graham (1933). It has been strengthened by Tippelskirch's (1956) experiment using liquid sulphur (the viscosity of which increases sharply with temperature in the range 153–190 °C while decreasing with temperature everywhere else) in which flow reversal was noted when the temperature exceeded 153 °C.

In bounded domains both 'upflow' and 'downflow' are possible and prediction of realizability for only one branch of convection does not apply. It is, therefore, an open question as to what should be anticipated in laboratory experiments. Some frequently cited experimental evidence for the realizability of just one form of convection is not completely convincing. Graham asserts without documentation the existence of only upflow in liquids, and he seems to have made no attempt to reach a second solution in his experiments with gases. Tippelskirch's experiments do not establish the non-existence of two stable forms of convection either. The flow reversal which is noted in this experiment would be

† Stability here is defined with respect to infinitesimal disturbances which are almost periodic (AP) in the variable x, y of the horizontal plane. Almost periodic functions are a considerable generalization of periodic functions which leave intact the property of completeness of the '*Fourier*' *series*. The AP functions are uniform limits of polynomials of exponentials of the form (Bohr 1932)

$$\sum_{-N}^{N} C_n \exp\{i(\alpha_n x + \beta_n y)\}. \tag{1.2}$$

Such polynomials are eigenfunctions for Laplace's operator in the plane with eigenvalue $\alpha_n^2+\beta_n^2 = a^2$. These eigenfunctions of the membrane equation then form the intersection of the plan-form functions (cf. footnote p. 259) and the AP functions. Not every plan-form function is AP. For example, the Bessel function $J_0(\lambda r)$ has $\Delta_2 J_0 + \lambda^2 J_0 = 0$ but is not AP. In fact, no function in $L_2(x, y)$ could be AP.

The stability studies of Schluter, Lortz & Busse (1965), Lortz (1961), Busse (1962) and Krishnamurti (1968) are to be understood in terms of the mathematics of the AP disturbances. At lowest order, the intersection of the AP functions and plan-form functions is considered. In generating the non-linear solution, the polynomials (1.2) lead always to yet larger polynomials, and in the limit $N \to \infty$, one has a 'Fourier' series which, if convergent, represents an almost periodic, and not necessarily periodic, solution of the problem. If $\mathscr{R}_c$ is simple and one seeks *periodic* solutions, then the process is convergent (Fife & Joseph 1969).

‡ Busse (1962) shows this when thermal properties vary. Krishnamurti (1968) shows it when there are heat sources. In a more restricted class of solutions, the same result is shown to hold by Palm & Øiann (1964), following earlier work of Palm (1960) and Segel & Stuart (1962) for the variable-viscosity problem, by Davis & Segel (1968) when there is a free surface and it is allowed to deflect, and by Scanlon & Segel (1967) when surface tension drives the convection.

expected from either one of the possible branches in the range of temperature in which the derivative of viscosity with respect to temperature changes sign (see figure 2).

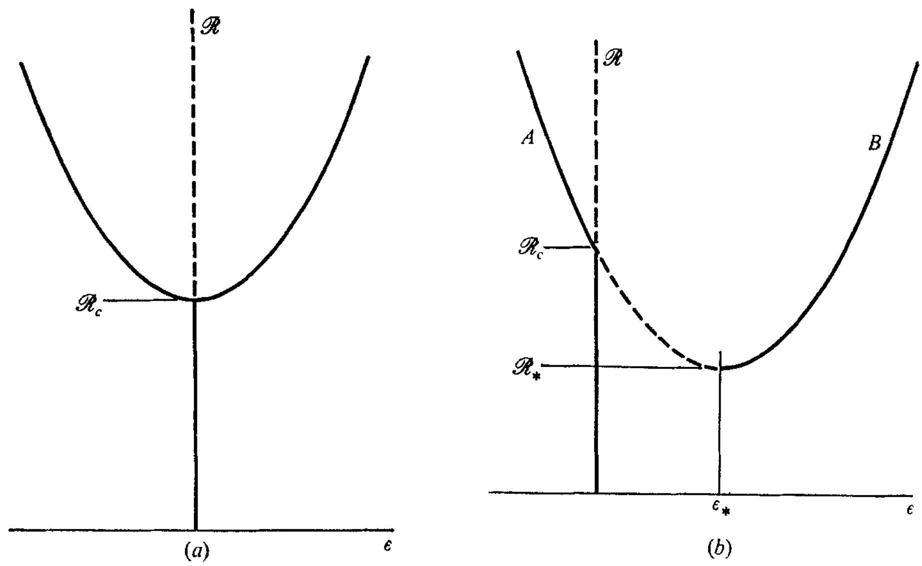

FIGURE 1. (*a*) Stability sketch for the 'Bénard' problem ($\xi^2+\zeta^2 = 0$) in the bounded domain when $\mathscr{R}_c$ is a simple eigenvalue. The conduction solution has $\epsilon = 0$. Possible steady convection is associated with curves $\mathscr{R}(\epsilon)$ shown in the figure as heavy lines. Solid lines show linearly stable solutions, and dashed lines give unstable solutions. In this problem there can be no subcritical convection (convection with $\mathscr{R}(\epsilon) < \mathscr{R}_c$). Both 'upflow' and 'downflow' are stable.

Three-dimensional convection in fluid layers is unstable (Schluter *et al.* 1965) when ($\xi^2+\zeta^2 = 0$), so that a closed cell analogy of the bounded domain situation does not exist. Convection in two-dimensional rolls however is stable. In this case, the two solutions (upflow and downflow) can be obtained from one another by horizontal shifting of the co-ordinates.

(*b*) Stability sketch for generalized convection when $\mathscr{R}_c(\xi, \zeta)$ is a simple eigenvalue and $\xi^2+\zeta^2 > 0$ is small. When $\mathscr{R} < \mathscr{R}_*$, there is no steady convection for small ϵ. When $\mathscr{R}_* < \mathscr{R} < \mathscr{R}_c$, there is steady linearly stable convection and linearly stable conduction. When $\mathscr{R} > \mathscr{R}_c$, conduction has exchanged its stability with the supercritical convective branch A, and this convection and that associated with the subcritical branch B are both linearly stable.

This picture differs slightly from the one which is appropriate to convection of hexagonal form in fluid layers. In the layer, $\mathscr{R}_c$ is not a simple eigenvalue, and roll eigenfunctions can (and do) destabilize the supercritical branch A. For the hexagonal convection, the branch A would be shown as a dotted line.

In any event, it is clear already from the analysis and experiment of Liang, Vidal & Acrivos (1969) that the view which was once accepted can no longer be asserted without qualification. These last-named authors show a photograph (figure 3 of their paper) of *two* stable forms of convection in a cylindrical container and give, also, a perturbation analysis which supports the notion that both branches are stable.

I wish to demonstrate that the result of Liang *et al.* (1969) is true for all the generalized problems mentioned at the start of this paper. The main hypothesis which is required is the simplicity of the eigenvalue $\mathscr{R}_c$. In this respect our analysis has something in common with (and is partly inspired by) a recent analysis of Sattinger (1970), who shows that subcritical branches are unstable and supercritical branches are stable, a result which superficially disagrees with all those mentioned so far, but which actually describes the stability picture shown in

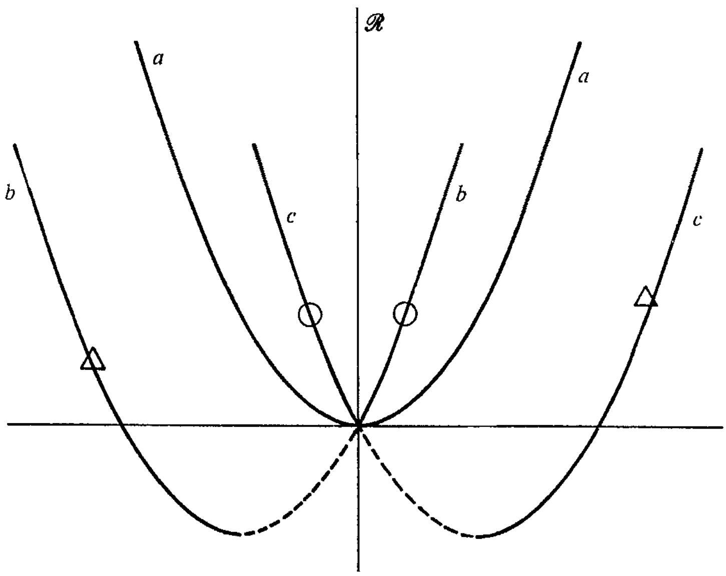

FIGURE 2. Schematic sketch of convection of liquid sulphur in a bounded domain when $\mathscr{R}_c$ is simple. The curve *aa* is for a mean temperature of 153 °C. At this temperature the rate of change of viscosity with temperature vanishes, and the up and down motions are both stable. The curves *bb* and *cc* are on this or that side of 153 °C, and the stability situation is like that shown in figure 1. Consider an experiment in which the mean temperature is allowed to vary. Suppose the conditions of the experiment favour observation of subcritical branches ($\triangle$). Then one could observe a change in sign ($\pm\epsilon$) of the convection as the mean temperature is varied across 153 °C. The same remark holds relative to supercritical branches $\bigcirc$. Observing $\triangle$ does not exclude or establish the existence of stable convection on branches $\bigcirc$.

figure 1 when $|\epsilon| < |\epsilon_*|$. The results which Sattinger obtains using the Leray–Schauder theory of the topological degree of a mapping are here obtained by direct perturbations (to order ϵ). For larger amplitudes ($O(\epsilon^2)$) the more direct analysis shows that the subcritical solutions which are unstable at order ϵ can regain stability and do regain stability when $\xi^2+\zeta^2$ is small.

The analysis also shows that the curvature of the viscosity function with respect to temperature, which is neglected when this relation (or the corresponding ones for other material functions) is linearized, enters in an important way in the formula for $\partial^2\mathscr{R}(0,\xi,\zeta)/\partial\epsilon^2$.

The main requirement of the stability results given in §4 is the simplicity of $\mathscr{R}_c$. I was unable to show anything very general about the multiplicity of $\mathscr{R}_c$ and was forced to treat instead (in §5) some very simple special cases of Bénard convection ($\xi = \zeta = 0$). One example, the 'free surface box', is just a period cell, but the boundary conditions on it are to be regarded as prescribed. Everything can be shown for this simple example. But in the course of developing the example, a solution was discovered for axisymmetric convection in a ring or disk when the top and bottom surfaces are free and the side wall is rigid. As far as I know, the solution is new. Some small interest accrues to the solution, because unlike the usual situation in which one must solve a sixth-order problem in the vertical (z) direction to satisfy conditions at the rigid boundary, here one must solve a somewhat different sixth-order problem in the horizontal (r) direction for just the same reason.

Section 6 deals with the following question: How can it come about that layers of large horizontal extent, in which $\mathscr{R}_c$ is simple, can act like infinite layers with regard to stability of the two possible forms of steady convection?

2. The branching problem and the linear stability problem for convection

The starting point is the system of O–B equations for disturbances $\mathbf{u}^*$ and $\theta^* = T^* - \tilde{T}^*(x_3)$ of the conduction solution of the problem

$$d^2\tilde{T}^*/dx_3^2 + \eta H(x_3) = 0.$$

Here $\eta H(x_3)$ is the heat source distribution, η is the source of strength, $x_3 = 0$ is the bottom of the container Ω, and the temperature of the bottom and top is

$$T_B^* = T_T^* + \Delta T \quad \text{and} \quad T_T^*,$$

respectively. The O–B equations are generalized to allow for a temperature-dependent kinematic viscosity of the form

$$\nu_0 \gamma\{\xi^*(T^*(x_i) - T_B^*)\} \quad (i = 1, 2, 3),$$

where ξ^* is a constant, γ is the viscosity function and $\nu = \nu_0$ at the bottom of the container ($\gamma(0) = 1$). Then

$$\partial \mathbf{u}^*/\partial t^* + \mathbf{u}^* . \nabla \mathbf{u}^* = -\nabla p^* + \alpha g \theta^* \mathbf{k} + 2\nu_0 \nabla . \gamma \mathbf{d}^*,$$
$$\partial \theta^*/\partial t^* + \mathbf{u}^* . \nabla \theta^* + w^* d\tilde{T}^*/dx_3 = \kappa \Delta \theta^*.$$

Here $\mathbf{k}$ is the unit vector in the direction of increasing x_3 (against gravity), $(\mathbf{d}^*)_{ij} = \frac{1}{2}(\partial_i u_j^* + \partial_j u_i^*)$, w^* is the x_3 component of velocity and κ, α, g are the thermal diffusivity, expansion coefficient and the magnitude of gravity, respectively.

Before generalizing the problem further, it will be convenient to introduce dimensionless variables $[t, \mathbf{x}, \mathbf{u}, \theta, p, \tilde{T}]$ by dividing $[t^*, \mathbf{x}^*, \mathbf{u}^*, \theta^*, p^*, \tilde{T}^*]$ by scale factors $[l^2/\nu_0, l, \nu_0/l, \Delta T \mathscr{P}/\mathscr{R}, \nu_0^2/l^2, \Delta T]$. Then

$$\partial \mathbf{u}/\partial t + \mathbf{u} . \nabla \mathbf{u} = -\nabla p + \mathscr{R}\theta \mathbf{k} + 2\nabla . \gamma \mathbf{d}, \tag{2.1a}$$

$$\mathscr{P}(\partial \theta/\partial t + \mathbf{u} . \nabla \theta) + \mathscr{R} w d\tilde{T}/dz = \Delta \theta, \quad \operatorname{div} \mathbf{u} = 0, \tag{2.1b, c}$$

where the conduction temperature (in dimensionless form) is

$$\tilde{T}(z) - T(0) = -\left(z + \xi \int_0^z g(z')\, dz'\right) = \hbar(z, \xi),$$

and
$$\gamma\{\xi^*(T^*-T_B^*)\} = \gamma\{\zeta[\mathscr{P}\theta/\mathscr{R}-\hbar(z,\xi)]\}. \tag{2.2}$$

Here the constant $\xi^*\Delta T = \zeta$ measures the rate of change of viscosity with temperature, $\mathscr{P} = \nu_0/\kappa$ is the Prandtl number, $\mathscr{R}^2 = \alpha g \Delta T l^3/\nu_0\kappa$ is the Rayleigh number, $T(0) = T_B^*/\Delta T$ and

$$\xi g(z) = d\tilde{T}(0)/dz + 1 - (l^2/\Delta T)\int_0^z \eta H(z')\,dz' \tag{2.3}$$

defines heat source parameter ξ. It is convenient to take $z = 0$ at the lowest point of the container Ω. Then in all the problems, the fluid has viscosity ν_0 at the bottom.

On the boundary $\partial\Omega$ of Ω, we require that

$$\mathbf{u}.\mathbf{n} = 0|_{\partial\Omega},\ \mathbf{u} = 0|_{S_u},\ \theta = 0|_{S_T}, \tag{2.1d,e,f}$$

and
$$\mathbf{d}.\mathbf{n}-\mathbf{n}(\mathbf{n}.\mathbf{d}.\mathbf{n}) = 0|_{\partial\Omega-S_u}, \tag{2.1g}$$
$$\partial\theta/\partial n + h_T\theta = 0|_{\partial\Omega-S_T},$$

where S_T and S_u are portions of $\partial\Omega$ (with outward normal $\mathbf{n}$), and $h_T(\mathbf{x})$ is a non-negative function defined on $\partial\Omega$. Equation (2.1g) is a free surface condition; it requires that tangential tractions vanish on $\partial\Omega - S_u$.

To shorten the writing it is useful to introduce a matrix notation. We first identify the non-linear part $\tau(\xi,\zeta,\theta)$ of the viscosity function

$$\gamma(\zeta\mathscr{P}\theta/\mathscr{R}-\zeta\hbar(z,\xi)) = \tau(\xi,\zeta;\theta)+\gamma(-\zeta\hbar(z,\xi)). \tag{2.4}$$

Hence, $\tau = 0$ when $\theta = 0$. Equations (2.1a,b,c) may be written as

$$\left(\frac{\partial}{\partial t}+\mathbf{u}.\nabla\right)\begin{bmatrix}u\\v\\w\\\mathscr{P}\theta\end{bmatrix} = -\begin{bmatrix}\partial_x p\\\partial_y p\\\partial_z p\\0\end{bmatrix}+\mathscr{R}\left[\begin{array}{ccc|c} & 0 & & 0\\ & & & 0\\ & & & 1\\\hline 0 & 0 & 1+\xi g(z) & 0\end{array}\right]\begin{bmatrix}u\\v\\w\\\theta\end{bmatrix}$$
$$+(\partial_x,\partial_y,\partial_z,0)\left(\left[\begin{array}{ccc|c} & & & 0\\ & 2\tau(\xi,\zeta;\theta)\,\mathbf{d} & & 0\\ & & & 0\\\hline 0 & 0 & 0 & 0\end{array}\right]+\left[\begin{array}{ccc|c} & & & \partial_x\theta\\ & 2\hat{\gamma}\mathbf{d} & & \partial_y\theta\\ & & & \partial_z\theta\\\hline 0 & 0 & 0 & 0\end{array}\right]\right), \tag{2.5a}$$

where $\hat{\gamma} \equiv \gamma(-\zeta\hbar(z,\xi)) > 0$ and $(\mathbf{d})_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. In matrix notation we may write

$$\partial\mathbf{Q}^\dagger/\partial t + (\mathbf{Q}.\partial)\mathbf{Q}^\dagger = -\boldsymbol{\partial} p + \mathscr{R}\mathscr{F}.\mathbf{Q} + \boldsymbol{\partial}.\{\boldsymbol{\tau}(\xi,\zeta;\theta);.\mathbf{Q}+\mathfrak{d};\mathbf{Q}\}, \tag{2.5b}$$

where the order in which terms appear in (2.5a) is preserved in (2.5b). Here

$$\mathbf{Q} = (u,v,w,\theta),\quad \mathbf{Q}^\dagger = (u,v,w,\mathscr{P}\theta),\ \boldsymbol{\partial} = (\partial_x,\partial_y,\partial_z,0),$$

$\mathscr{F}$ is the 4×4 matrix in (1.5a) which appears as the coefficient of $\mathscr{R}$, $(\mathfrak{d};\mathbf{Q})$ is the 4×4 matrix linear in the first derivatives of $\mathbf{Q}$ and defined by

$$\mathfrak{d};\mathbf{Q} = \left[\begin{array}{ccc|c} & & & \partial_x\theta\\ & 2\hat{\gamma}\mathbf{d} & & \partial_y\theta\\ & & & \partial_z\theta\\\hline 0 & 0 & 0 & 0\end{array}\right]$$

and $(\boldsymbol{\tau};\, \mathbf{Q})$ is the 4×4 matrix defined by

$$(\boldsymbol{\tau}(\xi,\zeta;\theta);\mathbf{Q}) = \left[\begin{array}{c|c} & 0 \\ 2\tau(\xi,\zeta;\theta)\,\mathbf{d} & 0 \\ & 0 \\ \hline 0 \quad 0 \quad 0 & 0 \end{array}\right].$$

This matrix is not linear in the components of $\mathbf{Q}$ because τ depends on θ. We note that $\mathbf{Q}^{\dagger} = \mathbf{I}^{\dagger}.\mathbf{Q}$ where $\mathbf{I}^{\dagger} = \operatorname{diag}(1,1,1\,\mathscr{P})$ is a 4×4 diagonal matrix.

The non-linear parts of (2.5*b*) are to be grouped together under the single term

$$\mathscr{B}[\xi,\zeta;\mathbf{Q}] = -\mathbf{Q}.\boldsymbol{\partial}\mathbf{Q}^{\dagger} + \boldsymbol{\partial}.\{\boldsymbol{\tau}(\xi,\zeta;\theta);\mathbf{Q}\}. \tag{2.5c}$$

The boundary conditions may be written as

$$(\mathbf{F}+\mathbf{B}).\mathbf{Q} = 0|_{\partial\Omega}. \tag{2.5d}$$

Here $\mathbf{N} = (n_x, n_y, n_z, 0)$ is the normal vector (outward on $\partial\Omega$) defined for 4 components, $\mathbf{F}$ is the linear operator defined by

$$\mathbf{F}.\mathbf{Q} = \mathbf{N}.\mathbf{Q}.\mathbf{N} + [\mathbf{N}.\mathfrak{d};\mathbf{Q} - \mathbf{N}\{\mathbf{N}.(\mathfrak{d};\mathbf{Q}).\mathbf{N}\}]$$

and $\mathbf{B}$ is a 4×4 diagonal matrix

$$\mathbf{B} = \operatorname{diag}(h_u, h_u, h_u, h_T),$$

where $h_u = \infty$ on S_u, $h_T = \infty$ on S_T and $h_u = 0|_{\partial\Omega - S_u}$. It may ease the work of the casual reader to replace (2.5*d*) with $\mathbf{Q} = 0|_{\partial\Omega}$ as in (2.12*c*).

In the work which follows, Ω is a bounded domain in three dimensions. Ω could be a period cell defined by periodic disturbances in a fluid layer. Integration of any quantity $f(x,y,z)$ over Ω is indicated by the angle bracket $\langle f\rangle$.

Steady convection is governed by the following set of equations:

$$\boldsymbol{\partial}.(\mathfrak{d};\mathbf{Q}) + \mathscr{R}\mathscr{F}.\mathbf{Q} + \mathscr{B}[\xi,\zeta;\mathbf{Q}] = \boldsymbol{\partial}p, \tag{2.6a}$$

$$\boldsymbol{\partial}.\mathbf{Q} = 0, \quad (\mathbf{F}+\mathbf{B}).\mathbf{Q} = 0|_{\partial\Omega}. \tag{2.6b, c}$$

Let $\tilde{\mathbf{Q}}$ be any vector which satisfies (2.6*b*, *c*). Then

$$\langle\tilde{\mathbf{Q}}.(\boldsymbol{\partial}.(\mathfrak{d};\mathbf{Q}) + \mathscr{R}\mathscr{F}.\mathbf{Q})\rangle = \langle\mathbf{Q}.(\boldsymbol{\partial}.(\mathfrak{d};\tilde{\mathbf{Q}}) + \mathscr{R}\mathscr{F}^{T}.\tilde{\mathbf{Q}})\rangle.$$

Here $\mathscr{F}^T$ is the transpose of $\mathscr{F}$. If $\mathscr{F}^T = \mathscr{F}$, then the linear problem for (2.6*a*, *b*, *c*), i.e. the problem with $\mathscr{B} = 0$, is formally self-adjoint. On the other hand,

$$\mathscr{F}^T - \mathscr{F} = \xi\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g(z) \\ 0 & 0 & -g(z) & 0 \end{bmatrix}. \tag{2.7}$$

Hence, ξ gives the deviation of the linear problem from self-adjointness. The parameter ζ is a measure of the energy-producing non-linearity. The combination of these two effects allows a large number of possible generalizations of the Boussinesq equations. For example, if the thermal diffusivity were allowed to be temperature dependent, then a term like ξ would arise in the diffusion term and, of course, the conduction solution could not then be linear in z (hence ξ).

It is necessary if $\mathscr{R}$ is not preassigned to assign a size to the non-linear solution and to seek the value $\mathscr{R}(\epsilon)$ as part of the non-linear solution. The size of the solution we seek is specified by the requirement that

$$\epsilon^2 = \langle \mathbf{Q}.\mathbf{Q}^\dagger \rangle. \quad (2.6d)$$

It is physically more natural to give $\mathscr{R}$ and find ϵ but here, in the mathematical problem we go the other way.

The system $(2.6a, b, c, d)$ has just two $(\pm\epsilon)$ solutions which branch off the conduction solution at the value $\mathscr{R}(0) \equiv \mathscr{R}(0, \xi, \zeta) = \mathscr{R}_c$. As has been mentioned these correspond to 'upflow' or 'downflow' at some given point in Ω. The solution can be constructed as a Taylor series in ϵ. To obtain results to order ϵ^2, it is necessary to retain the curvature of the viscosity function as well as its slope, but, without loss of generality, one can drop higher-order terms and use in $\mathscr{B}$ a truncated Taylor series. Let $\gamma(\zeta\mathscr{P}\theta/\mathscr{R} - \zeta h) = \gamma(x)$. Then, from (2.4) one finds that

where $$\tau(\xi,\zeta;\theta) = (\mathscr{P}\zeta\theta/\mathscr{R})\gamma' + \tfrac{1}{2}(\mathscr{P}\zeta\theta/\mathscr{R})^2\gamma'',$$
$$\gamma' = \gamma'(\xi,\zeta,z), \gamma'' = \gamma''(\xi,\zeta,z)$$

are derivatives with respect to x evaluated at $\theta = 0$, that is, $x = -\zeta h(z,\xi)$.

Then, $\mathscr{B}$ may be written as

$$\mathscr{B}[\xi,\zeta;\mathbf{Q}] = \mathbf{T}[\xi,\zeta;\mathbf{Q},\mathbf{Q}] + \mathbf{G}[\xi,\zeta;\mathbf{Q},\mathbf{Q},\mathbf{Q}],$$

where $$\mathbf{T}[\xi,\zeta;\mathbf{Q},\mathbf{Q}] = -\mathbf{Q}.\boldsymbol{\partial}.\mathbf{Q} - (\mathscr{P}/\mathscr{R})\zeta\boldsymbol{\partial}.\{\theta\gamma'(\xi,\zeta,z)\,\mathbf{d};\mathbf{Q}\}, \quad (2.8a)$$

and $$\mathbf{G}(\xi,\zeta;\mathbf{Q},\mathbf{Q},\mathbf{Q}] = (\mathscr{P}^2\zeta^2/\mathscr{R}^2)\boldsymbol{\partial}.\{\theta^2\tfrac{1}{2}\gamma''(\xi,\zeta,z)\,\mathbf{d};\mathbf{Q}\}. \quad (2.8b)$$

To study the linear stability of the convective branches $\mathscr{R}(\epsilon,\xi,\zeta)$, $\mathbf{Q}(\mathbf{x},\epsilon,\xi,\zeta)$, one studies the spectral problem

$$\boldsymbol{\partial}.(\mathbf{d};\mathbf{v}) + \mathscr{R}\mathscr{F}.\mathbf{v} - \sigma\mathbf{I}^\dagger.\mathbf{v} + \mathscr{B}[\xi,\zeta;\mathbf{Q};\mathbf{v}] = \boldsymbol{\partial}\pi, \quad (2.9a)$$
$$\boldsymbol{\partial}.\mathbf{v} = 0, \quad (\mathbf{F}+\mathbf{B}).\mathbf{v} = 0|_{\partial\Omega}, \quad 1 = \langle \mathbf{v}^\dagger.\mathbf{v}\rangle, \quad (2.9b,c,d)$$

where $\mathbf{v}$ is an infinitesimal disturbance of $\mathbf{Q}$, and

$$\mathscr{B}[\xi,\zeta;\mathbf{Q};\mathbf{v}] = \mathbf{T}[(\mathbf{Q},\mathbf{v}) + (\mathbf{v},\mathbf{Q})] + \mathbf{G}[(\mathbf{Q},\mathbf{Q},\mathbf{v}) + (\mathbf{Q},\mathbf{v},\mathbf{Q}) + (\mathbf{v},\mathbf{Q},\mathbf{Q})].$$

Here and below, the parameters ξ and ζ have been suppressed in the notation, but they are understood. The linear stability of the convection $\mathbf{Q}$ is determined by the sign of Re (σ) (Re means real part and Re $(\sigma) > 0$ means instability).

It proves convenient to set $\mathbf{Q} = \epsilon\mathbf{q}$ and to speak of branching solutions of

$$\boldsymbol{\partial}.(\mathbf{d};\mathbf{q}) + \mathscr{R}\mathscr{F}.\mathbf{q} + \epsilon\mathbf{T}[\mathbf{q},\mathbf{q}] + \epsilon^2\mathbf{G}[\mathbf{q},\mathbf{q},\mathbf{q}] = \boldsymbol{\partial}p \quad (2.10a)$$

and $$\boldsymbol{\partial}.\mathbf{q} = 0, \quad (\mathbf{F}+\mathbf{B}).\mathbf{q} = 0|_{\partial\Omega}, \quad \langle \mathbf{q}.\mathbf{q}^\dagger\rangle = 1. \quad (2.10b,c,d)$$

The stability of the two convection solutions of (2.10) is governed by

$$\boldsymbol{\partial}.(\mathbf{d};\mathbf{v}) + \mathscr{R}\mathscr{F}.\mathbf{v} - \sigma\mathbf{I}^\dagger.\mathbf{v} + \epsilon\mathbf{T}[(\mathbf{q},\mathbf{v}) + (\mathbf{v},\mathbf{q})]$$
$$+ \epsilon^2\mathbf{G}[(\mathbf{q},\mathbf{q},\mathbf{v}) + (\mathbf{q},\mathbf{v},\mathbf{q}) + (\mathbf{v},\mathbf{q},\mathbf{q})] = \boldsymbol{\partial}\pi. \quad (2.11)$$

It will simplify the writing a little and allow us to use some of the known results of analytic perturbation theory to replace the conditions $(2.9b, c, d)$ with

$$\boldsymbol{\partial}.\mathbf{v} = 0, \quad \mathbf{v} = 0|_{\partial\Omega}, \quad \langle \mathbf{v}.\mathbf{v}^\dagger\rangle = 1. \quad (2.12b,c,d)$$

When (2.12*b*) holds for $\mathbf{v}$ it also holds for $\mathbf{q}$. The formal part of the analysis given below, however, could be as easily constructed relative to the natural boundary conditions (2.10*c*). It will also be evident that the results given hold for $\mathbf{T}$ and $\mathbf{G}$ of a general form and not just for the operators defined by (2.8*a*, *b*).

The use of analytic perturbation theory for (2.10*a*) and (2.12*b*, *c*, *d*) is justified in the papers of Fife & Joseph (1969) and Fife (1970). The analyticity of the solution $\mathbf{v}(\epsilon, \xi, \zeta; \mathbf{x})$ and $\sigma(\epsilon, \xi, \zeta)$ in three parameters could be proved by the method of dominating majorants as in the first of the above-mentioned papers.† Alternately, one could follow Kirchgässner & Sorger (1968) (hereafter referred to as K & S) and apply analytic perturbation theory.

3. The linear stability problem for conduction

Let $\epsilon = 0$ in (2.11*a*). Then $\mathscr{R}$ can be preassigned and we seek eigenvalues $\sigma(\mathscr{R}, \xi, \zeta)$ of the problem

$$\mathscr{M}.\mathbf{v} \equiv \partial.(\mathfrak{d};\mathbf{v}) + \mathscr{R}\mathscr{F}.\mathbf{v} - \sigma\mathbf{I}^{\dagger}.\mathbf{v} = \partial\pi \tag{3.1}$$

and (2.12*b*, *c*, *d*). This problem governs the stability of the conduction solution. When $\mathscr{R}$ is small enough, $\operatorname{Re}(\sigma) < 0$ for all eigenvalues of (3.1). The neutral limit is the value $\mathscr{R} = \mathscr{R}_c(\xi, \zeta)$, for which the eigenvalue with the largest real part just vanishes $\operatorname{Re}(\sigma(\mathscr{R}_c, \xi, \zeta)) = 0$. The main assumptions of our analysis concern the values of $\sigma(\mathscr{R}_c, \xi, \zeta)$. These are:

A:
(1) $\mathscr{R}_c$ is an algebraically simple eigenvalue of (3.1) when $\sigma = 0$ and (ξ, ζ) lies in a closed region Γ containing $(0, 0)$.
(2) A principle of exchange of stability (PES) holds in the sense that $\operatorname{Im}(\sigma) = 0$ when $\operatorname{Re}(\sigma) \geqslant -a^2$ for some $a^2 > 0$.

The assumption (2) is not necessary when $\xi = 0$, for then the problem (3.1) and (2.12*b*, *c*, *d*) are self-adjoint (cf. (2.7)), and the entire spectrum $\Sigma\sigma(\mathscr{R}, \xi, \zeta)$ is real. But even if $\mathscr{M}$ of (3.1) is not a self-adjoint operator, $\Sigma\sigma(\mathscr{R}, \xi, \zeta)$ is composed entirely of discrete eigenvalues of finite multiplicity (Yudovich 1965, K & S 1968). Since $\mathscr{R}_c$ is a simple eigenvalue, $\sigma(\mathscr{R}_c, \xi, \zeta) = 0$ is also simple, and, by (2), $\sigma(\mathscr{R}_c, \xi, \zeta) = 0$ is the eigenvalue with the largest real part.

When $\xi \neq 0$ we will need the adjoint problem

$$\partial.(\mathfrak{d};\mathbf{v}^A) + \mathscr{R}\mathscr{F}^T.\mathbf{v}^A - \sigma\mathbf{I}^{\dagger}.\mathbf{v}^A = \partial\pi^A, \tag{3.2}$$

where $\mathbf{v}^A$ like $\mathbf{v}$ satisfies (2.12*b*, *c*, *d*). We will also use the following result, which extends only slightly results already obtained by K & S (1968):

† Here, we take analyticity in parameters as given. The interested reader can check the applicability of the theory of analytic operators of type A (Kato 1966, p. 375) to the problem (2.11) and (2.12*b*, *c*, *d*). It may help to note that when $\zeta = 0$, $\partial.(\mathfrak{d};\mathbf{v}) = \Delta\mathbf{v}$ and the problem does not differ in any mathematical essential from the one considered by K & S (1968). In particular our analogue of their operator $\tilde{A}$ would have the same domain and range as $\tilde{A}$ and would be closable. These properties would be shared by our analogue if $\zeta \neq 0$ and $\epsilon = 0$. Then $\partial.(\mathfrak{d};\mathbf{v})$ differs from $\gamma\Delta\mathbf{v}$ by terms with first derivatives only. The relevant properties of $\tilde{A}$ could be shown to persist even when $\zeta \neq 0$ and $\epsilon \neq 0$ but then it would require an additional argument. For example, one could use analyticity of $\mathscr{R}$ and $\mathbf{q}$ in ϵ and the uniform boundedness (in $\mathbf{x}$) of the Taylor coefficients $\mathbf{q}_l$ to establish the estimate of theorem (2.6) of Kato (1966, p. 377).

LEMMA 1. *Every eigenvalue σ of* (3.1) *and* (2.12b, c, d) *satisfies the relation*

$$\mathscr{R}^2 \frac{\partial(\sigma/\mathscr{R})}{\partial \mathscr{R}} \langle \mathbf{v}^A . \mathbf{v} \rangle = \langle \hat{\gamma} \mathbf{d} : \mathbf{d}^A + \nabla\theta . \nabla\theta^A \rangle. \tag{3.3}$$

Let conditions A hold. Then when $\mathscr{R} = \mathscr{R}_c$,

$$\partial\sigma/\partial\mathscr{R} > 0,$$

and $\sigma(\mathscr{R}, \xi, \zeta)$ is real when $|\mathscr{R} - \mathscr{R}_c|$ is sufficiently small.

Proof of Lemma 1

Since $\sigma = 0$ is simple when $\mathscr{R} = \mathscr{R}_c$ and $(\xi, \zeta)\epsilon\Gamma$, analytic perturbation theory guarantees that $\sigma(\mathscr{R}, \xi, \zeta)$ is analytic in $\mathscr{R}, \xi$ and ζ when $|\mathscr{R} - \mathscr{R}_c|$ is small and $(\xi, \zeta)\epsilon\Gamma$. An identical consideration shows that the eigenvalue $\mathscr{R}_c(\xi, \zeta)$ and eigenfunctions $\mathbf{v}$ and $\mathbf{v}^A$ are analytic functions of (ξ, ζ) in Γ.

It follows from $\mathscr{R}$ analyticity that $\partial\sigma/\partial\mathscr{R}$ and $\partial\mathbf{v}/\partial\mathscr{R}$ exist and must satisfy the problem obtained by differentiating (3.1) and (2.12b, c, d) with respect to $\mathscr{R}$. Thus,

$$\mathscr{M} . (\partial\mathbf{v}/\partial\mathscr{R}) = -\mathscr{F} . \mathbf{v} + (\partial\sigma/\partial\mathscr{R})\mathbf{I}^{\dagger} . \mathbf{v} \tag{3.4}$$

and $\partial\mathbf{v}/\partial\mathscr{R}$ must satisfy (2.12b, c, d). The right side of (3.4) must be orthogonal to solutions of the homogeneous adjoint equation (3.2). Then using (3.1), we find

$$\frac{\partial\sigma}{\partial\mathscr{R}} \langle \mathbf{v}^A . \mathbf{v}^{\dagger} \rangle = \langle \mathbf{v}^A . \mathscr{F} . \mathbf{v} \rangle = \frac{\sigma}{\mathscr{R}} \langle \mathbf{v}^A . \mathbf{v}^{\dagger} \rangle - \frac{1}{\mathscr{R}} \langle \mathbf{v}^A . \mathfrak{d}(\mathfrak{d}; \mathbf{v}) \rangle \tag{3.5}$$

and (3.3) follows by integration.

When $\sigma = 0$, since $\mathscr{R}_c$ is simple

$$-\langle \mathbf{v}^A . \mathfrak{d}(\mathfrak{d}; \mathbf{v}) \rangle = \langle \hat{\gamma} \mathbf{d} : \mathbf{d}^A + \nabla\theta . \nabla\theta^A \rangle \neq 0. \tag{3.6}$$

Otherwise $\langle \mathbf{v}^A . \mathscr{F} . \mathbf{v} \rangle = 0$, contradicting simplicity. When $\xi = 0$, (3.1) and (2.12b, c, d) form a self-adjoint problem and $\mathbf{v} = \mathbf{v}^A$. Then, when $\xi = 0$,

$$-\langle \mathbf{v}^A . \mathfrak{d}(\mathfrak{d}; \mathbf{v}) \rangle > 0.$$

Let $\mathscr{R} = \mathscr{R}_c$. Then $\sigma = 0$ and

$$\frac{\partial\sigma}{\partial\mathscr{R}} = -\frac{\langle \mathbf{v}^A . \mathfrak{d}(\mathfrak{d}; \mathbf{v}) \rangle}{\mathscr{R} \langle \mathbf{v}^A . \mathbf{v}^{\dagger} \rangle} > 0 \tag{3.7}$$

in Γ. To check this we first recall that the scalar products in equation (2.7) are analytic when $(\xi, \zeta)\epsilon\Gamma$ and $\mathscr{R} = \mathscr{R}_c$. Moreover, $\langle \mathbf{v}^A . \mathbf{v}^{\dagger} \rangle \neq 0$ in Γ since by the analyticity of σ, $\partial\sigma/\partial\mathscr{R}$ is finite for $\mathscr{R} = \mathscr{R}_c$ and $(\xi, \zeta)\epsilon\Gamma$. The inequality is clearly true when $\xi = 0$. Since the scalar products in (3.7) cannot change sign as (ξ, ζ) take on values in Γ, the inequality must also hold in Γ.

The conduction solution is linearly stable when $\mathscr{R} < \mathscr{R}_c$. When $\mathscr{R} = \mathscr{R}_c$, the growth rate $\sigma = 0$ is a maximum (assuming PES), and since $\mathscr{R}_c$ is simple, $\partial\sigma/\partial\mathscr{R} > 0$ when $\mathscr{R} = \mathscr{R}_c$. This shows that the conduction solution loses its stability when $\mathscr{R} > \mathscr{R}_c$.

4. Perturbation analysis for convection

Just as in the case of conduction, there is a 'growth rate' curve

$$\sigma\{\mathscr{R}(\epsilon, \xi, \zeta), \epsilon, \xi, \zeta\} = \sigma(\epsilon)$$

for infinitesimal disturbances of convection. This curve passes through the point of maximum growth rate ($\sigma = 0$ when $\mathscr{R} = \mathscr{R}_c$) for infinitesimal disturbances of the conduction solution. The growth rate $\sigma(\epsilon)$ appears as an eigenvalue of (2.11), and we study how it changes with ϵ by analytic perturbations. The conditions A guarantee that $\mathrm{Re}(\sigma) = \sigma = 0$ is the eigenvalue with the largest growth rate when $\mathscr{R} = \mathscr{R}_c$. The simplicity of $\sigma = 0$ means that only one eigenvalue branch $\sigma(\epsilon)$ can pass through the origin of the $(\sigma, \mathscr{R} - \mathscr{R}_c)$ plane (figure 3) and in a small circle around the origin only this branch $\sigma(\epsilon)$ is available to destabilize convection. Originally stable eigenvalues with $\mathrm{Re}\{\sigma(0)\} < -a^2$ could perturb with ϵ into unstable eigenvalues with $\mathrm{Re}\{\sigma(\epsilon)\} > 0$ but not if ϵ is sufficiently small (see figure 3).

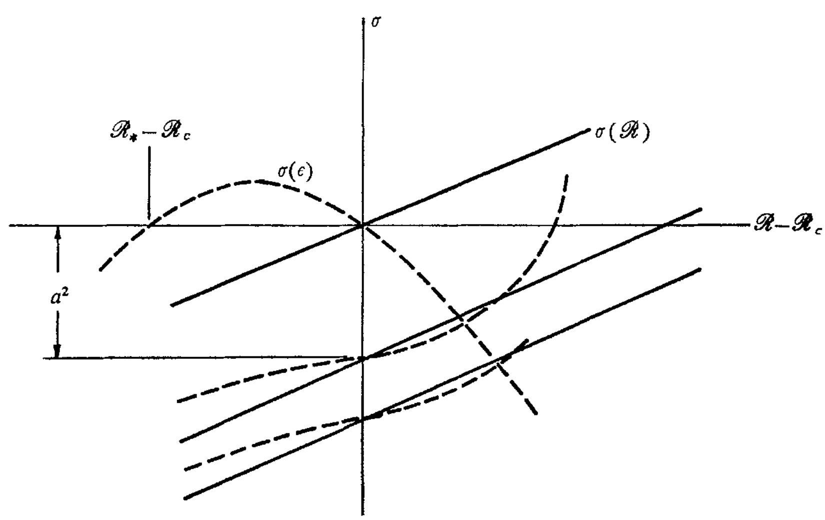

FIGURE 3. A schematic sketch of the growth rate curves for conduction (solid) and convection (dashed) lines. Sketched in the figure are what might be the appearance of the three largest eigenvalues for conduction $\sigma(\mathscr{R})$ and convection $\sigma(\epsilon)$. By PES the highest growth rate when $\mathscr{R} = \mathscr{R}_c$ is $\sigma = 0$. Since $\mathscr{R}_c$ is simple, $\sigma = 0$ is simple and the growth rate $\sigma(\mathscr{R})$ for conduction and $\sigma(\epsilon)$ for convection both pass through the origin. The subcritical convection $\mathscr{R}(\epsilon) - \mathscr{R}_c < 0$ is unstable when $\sigma(\epsilon) > 0$, but when $\epsilon > \epsilon_*$, $\sigma(\epsilon) < 0$ and the subcritical convection is stable. As the bounded domain is stretched into a layer, the distance between successive eigenvalues becomes smaller ($a^2 \to 0$), and stable eigenvalues $\sigma(0) \leqslant -a^2$ can then cross to $\sigma(\epsilon) \geqslant 0$ and become available to destabilize convection.

Now we undertake the construction of the perturbation solution up to terms of order ϵ^2. It should be noted, from the way in which the convection problem (2.10) and its stability counterpart (2.11) were formed, that the solutions of these two problems necessarily coincide with the solutions of (2.6) and (2.9), respectively, up to order ϵ^2.

The main result of this paper, theorem 1, follows from the formulas which are given in lemma 2, below. These formulas show how the Taylor coefficients (σ_1, σ_2) in the expansion

$$\sigma(\epsilon, \xi, \zeta) = \sigma_1(\xi, \zeta)\epsilon + \sigma_2(\xi, \zeta)\epsilon^2 + O(\epsilon^3) \tag{4.1}$$

of the largest eigenvalue of the problem (2.11*a*) and (2.12*b, c, d*) governing the linear stability of steady convection are related to the Taylor coefficients $(\mathscr{R}_1, \mathscr{R}_2)$ in the series representation

$$\mathscr{R}(\epsilon, \xi, \zeta) = \mathscr{R}_c(\xi, \zeta) + \mathscr{R}_1(\xi, \zeta)\epsilon + \mathscr{R}_2(\xi, \zeta)\epsilon^2 + O(\epsilon^3) \tag{4.2}$$

of the dimensionless temperature contrast necessary to drive steady convection.

Throughout §4 we shall designate ϵ derivatives evaluated at $\epsilon = 0$ with a number subscript, e.g. $2\mathbf{q}_2 = \partial^2\mathbf{q}(\epsilon, \xi, \zeta; x)/\partial\epsilon^2|_{\epsilon=0}$. In the remainder of §4,

$$\mathbf{q} = \mathbf{q}(\epsilon, \xi, \zeta; \mathbf{x})|_{\epsilon=0}$$

with an identical convention for $\mathbf{v}$.

LEMMA 2. *Let conditions A hold. Then*

$$\langle \mathbf{q}^A . \mathbf{q}^\dagger \rangle \equiv b^2 > 0,$$

$$\sigma_1 b^2 + \mathscr{R}_1 = 0, \tag{4.3}$$

and

$$\sigma_2 b^2 + 2\mathscr{R}_2 + \mathscr{R}_1 \phi = 0, \tag{4.4}$$

where ϕ is the bounded function of $(\xi, \zeta)\epsilon\Gamma$ defined by (4.19).

Proof of Lemma 2

Let $\mathscr{R} = \mathscr{R}_c$. Then $\epsilon = 0$ and $\sigma = 0$. Compare (2.10) and (2.11) and, on taking account of the simplicity of $\mathscr{R}_c$, conclude that $\mathbf{v} = \beta\mathbf{q}$. Similarly, $\mathbf{v}^A = \beta'\mathbf{q}^A$. We may normalize $\mathbf{v}^A$ and $\mathbf{q}^A$ so that

$$\langle \mathbf{q}^A . \mathscr{F} . \mathbf{q} \rangle = \langle \mathbf{v}^A . \mathscr{F} . \mathbf{v} \rangle = 1.$$

This equation and the condition $\langle \mathbf{v} . \mathbf{v}^\dagger \rangle = \langle \mathbf{q} . \mathbf{q}^\dagger \rangle = 1$ implies that $\beta = \beta' = \pm 1$.

The perturbation formulas to determine the Taylor coefficients to order ϵ^2 can be obtained from (2.10*a*) (2.11) and (2.12*b, c, d*) by direct differentiation with respect to ϵ or by substitution into these equations of the series (4.1, 4.2),

$$\mathbf{q}(\epsilon, \xi, \zeta; \mathbf{x}) = \mathbf{q} + \mathbf{q}_1\epsilon + \mathbf{q}_2\epsilon^2 + \ldots \tag{4.5a}$$

and

$$\mathbf{v}(\epsilon, \xi, \zeta; \mathbf{x}) = \mathbf{v} + \mathbf{v}_1\epsilon + \mathbf{v}_2\epsilon^2 + \ldots. \tag{4.5b}$$

For the steady convection, one finds that

$$\mathscr{R}_1\mathscr{F} . \mathbf{q} + \boldsymbol{\partial} . (\mathbf{\mathfrak{d}}; \mathbf{q}_1) + \mathscr{R}_c\mathscr{F} . \mathbf{q}_1 + \mathbf{T}[\mathbf{q}, \mathbf{q}] = \boldsymbol{\partial} p_1, \tag{4.6}$$

$$\mathscr{R}_2\mathscr{F} . \mathbf{q} + \mathscr{R}_1\mathscr{F} . \mathbf{q}_1 + \boldsymbol{\partial} . (\mathbf{\mathfrak{d}}; \mathbf{q}_2) + \mathscr{R}_c\mathscr{F} . \mathbf{q}_2 + \mathbf{T}[(\mathbf{q}_1, \mathbf{q}) + (\mathbf{q}, \mathbf{q}_1)] + \mathbf{G}[\mathbf{q}, \mathbf{q}, \mathbf{q}] = \boldsymbol{\partial} p_2. \tag{4.7}$$

with

$$0 = \boldsymbol{\partial} . \mathbf{q}_1 = \boldsymbol{\partial} . \mathbf{q}_2 = \mathbf{q}_1|_{\partial\Omega} = \mathbf{q}_2|_{\partial\Omega} = \langle \mathbf{q}_1 . \mathbf{q}^\dagger \rangle = \langle \mathbf{q}_2 . \mathbf{q}^\dagger + \mathbf{q}_2 . \mathbf{q}^\dagger + \mathbf{q}_1 . \mathbf{q}_1^\dagger \rangle. \tag{4.8}$$

The governing equations for the stability problem (σ and $\mathbf{v}$) are found in the same way. Conditions (4.8) hold for $\mathbf{v}$. Introducing $\hat{\mathbf{v}} = \mathbf{v}/\beta = \mathbf{q}$ (but, of course, in general $\hat{\mathbf{v}}_i = \mathbf{v}_i/\beta \neq \mathbf{q}_i, i = 1, 2$) one finds the perturbation equations for the stability problem in the form

$$\mathscr{M}_1 . \mathbf{q} + \partial . (\mathfrak{d} . \hat{\mathbf{v}}_1) + \mathscr{R}_c \mathscr{F} . \hat{\mathbf{v}}_1 + 2\mathbf{T}[\mathbf{q}, \mathbf{q}] = \partial \hat{\pi}_1 \tag{4.9}$$

and

$$\mathscr{M}_2 . \mathbf{q} + \mathscr{M}_1 . \hat{\mathbf{v}}_1 + \partial . (\mathfrak{d}; \hat{\mathbf{v}}_2) + \mathscr{R}_c(\mathscr{F} . \hat{\mathbf{v}}_2) + \mathbf{T}[(\mathbf{q}_1, \mathbf{q}) + (\mathbf{q}, \hat{\mathbf{v}}_1) + (\hat{\mathbf{v}}_1, \mathbf{q}) + (\mathbf{q}, \mathbf{q}_1)] + 3\mathbf{G}[\mathbf{q}, \mathbf{q}, \mathbf{q}] = \partial \hat{\pi}_2. \tag{4.10}$$

$\mathscr{M}$ is defined by (3.1) with $\mathscr{R} = \mathscr{R}_c$ and $\sigma = 0$, $\mathscr{M}_1 = \mathscr{R}_1 \mathscr{F} - \sigma_1 \mathbf{I}$ and

$$\mathscr{M}_2 = \mathscr{R}_2 \mathscr{F} - \sigma_2 \mathbf{I}^{\dagger}.$$

Equation (4.6)–(4.10) have unique solutions if the inhomogeneous terms are orthogonal to the eigenfunction of the adjoint problem (3.2) with $\mathscr{R} = \mathscr{R}_c(\sigma = 0)$. Therefore, we find from (4.6) that

$$\mathscr{R}_1 \langle \mathbf{q}^A . \mathscr{F} . \mathbf{q} \rangle + \langle \mathbf{q}^A . \mathbf{T}[\mathbf{q}, \mathbf{q}] \rangle = 0; \tag{4.11}$$

from (4.9) that

$$\mathscr{R}_1 \langle \mathbf{q}^A . \mathscr{F} . \mathbf{q} \rangle - \sigma_1 b^2 + 2\langle \mathbf{q}^A . \mathbf{T}[\mathbf{q}, \mathbf{q}] \rangle = 0; \tag{4.12}$$

from (4.7) that

$$\mathscr{R}_2 \langle \mathbf{q}^A . \mathscr{F} . \mathbf{q} \rangle + \mathscr{R}_1 < \mathbf{q}^A . \mathscr{F} . \mathbf{q}_1 \rangle + \langle \mathbf{q}^A . \mathbf{T}[(\mathbf{q}_1, \mathbf{q}) + (\mathbf{q}, \mathbf{q}_1)] \rangle + \langle \mathbf{q}^A . \mathbf{G}[\mathbf{q}, \mathbf{q}, \mathbf{q}] \rangle = 0, \tag{4.13}$$

and from (4.10) that

$$\mathscr{R}_2 \langle \mathbf{q}^A . \mathscr{F} . \mathbf{q} \rangle - \sigma_2 b^2 + \mathscr{R}_1 \langle \mathbf{q}^A . \mathscr{F} . \hat{\mathbf{v}}_1 \rangle - \sigma_1 \langle \mathbf{q}^A . \hat{\mathbf{v}}_1^{\dagger} \rangle + \langle \mathbf{q}^A . \hat{\mathbf{T}} \rangle + 3\langle \mathbf{q}^A . \mathbf{G}[\mathbf{q}, \mathbf{q}, \mathbf{q}] \rangle = 0, \tag{4.14}$$

where

$$\hat{\mathbf{T}} \equiv \mathbf{T}[(\mathbf{q}_1, \mathbf{q}) + (\mathbf{q}, \mathbf{q}_1) + (\mathbf{q}, \hat{\mathbf{v}}_1) + (\hat{\mathbf{v}}_1, \mathbf{q})].$$

Equations (4.11) and (4.12) combine to give equation (4.3) of the lemma.

To proceed further we will need to show that

$$\hat{\mathbf{v}}_1 = 2\mathbf{q}_1 + \mathscr{R}_1 \boldsymbol{\psi}, \tag{4.15}$$

where $\mathscr{R}_1 \boldsymbol{\psi}$ tends to zero with $\mathscr{R}_1$. To prove (4.15) we form the equation

$$\partial . (\mathfrak{d}; [\hat{\mathbf{v}}_1 - 2\mathbf{q}_1)] + \mathscr{R}_c \mathscr{F} . [\hat{\mathbf{v}}_1 - 2\mathbf{q}_1] - \mathscr{R}_1 \mathscr{F} . \mathbf{q} - \sigma_1 \mathbf{I}^{\dagger} . \mathbf{q} = \partial(\hat{\pi}_1 - 2p_1) \tag{4.16}$$

by subtracting twice (4.6) from (4.9). Equation (4.16) has a unique solution $\mathbf{V} = \hat{\mathbf{v}}_1 - 2\mathbf{q}_1$ which is solenoidal, satisfies the boundary conditions if and only if

$$0 = \langle \mathbf{q}^A . (\mathscr{R}_1 \mathscr{F} . \mathbf{q} + \sigma_1 \mathbf{q}^{\dagger}) \rangle.$$

This is just formula (4.3). It allows one to set $\sigma_1 = -\mathscr{R}_1/b^2$ and shows that $\mathbf{V}$ scales with $\mathscr{R}_1$, i.e. $\mathbf{V} = \mathscr{R}_1 \boldsymbol{\psi}$ where $\boldsymbol{\psi}$ is the unique solenoidal solution of the problem

$$\partial . (\mathfrak{d}; \boldsymbol{\psi}) + \mathscr{R}_c \mathscr{F} . \boldsymbol{\psi} - \{\mathscr{F} . \mathbf{q} - \mathbf{q}^{\dagger}/b^2\} = \partial \pi'$$

and $\boldsymbol{\psi} = 0|_{\partial\Omega}$. This proves (4.15).

Using (4.15) we find that

$$\langle \mathbf{q}^A . \hat{\mathbf{T}} \rangle = 3\langle \mathbf{q}^A . \mathbf{T}[(\mathbf{q}_1, \mathbf{q}) + (\mathbf{q}, \mathbf{q}_1)] \rangle + \mathscr{R}_1 \langle \mathbf{q}^A . \mathbf{T}[(\mathbf{q}, \boldsymbol{\psi}) + (\boldsymbol{\psi}, \mathbf{q})] \rangle, \tag{4.17}$$

which when combined with (4.13) gives

$$\langle \mathbf{q}^A.\tilde{\mathbf{T}}\rangle = -3\mathscr{R}_2\langle \mathbf{q}^A.\mathscr{F}.\mathbf{q}\rangle - 3\mathscr{R}_1\langle \mathbf{q}^A.\mathscr{F}.\mathbf{q}_1\rangle - 3\langle \mathbf{q}^A.\mathbf{G}\rangle + \mathscr{R}_1\langle \mathbf{q}^A.\mathbf{T}[(\mathbf{q},\boldsymbol{\psi})+(\boldsymbol{\psi},\mathbf{q})]\rangle. \quad (4.18)$$

Equation (4.18) is now used to eliminate $\langle \mathbf{q}^A.(\tilde{\mathbf{T}}+3\mathbf{G})\rangle$ from (4.14). This leads to (4.4) with

$$\begin{aligned}\phi &= \langle \mathbf{q}^A.\{3\mathscr{F}.\mathbf{q}_1 - \mathscr{F}.\hat{\mathbf{v}}_1 - \mathbf{T}[(\mathbf{q},\boldsymbol{\psi})+(\boldsymbol{\psi},\mathbf{q})] + (\sigma_1/\mathscr{R}_1)\hat{\mathbf{v}}_1^\dagger\}\rangle \\ &= \langle \mathbf{q}^A\{\mathscr{F}.\mathbf{q}_1 - 2\mathbf{q}_1^\dagger/b^2 - \mathbf{T}[(\mathbf{q},\boldsymbol{\psi})+(\boldsymbol{\psi},\mathbf{q})] - \mathscr{R}_1\boldsymbol{\psi} - \mathscr{R}_1\boldsymbol{\psi}^\dagger/b^2\}\rangle \quad (4.19)\end{aligned}$$

proving lemma 2.

The theorem proved below concerns the stability of the two forms of convection. The two forms correspond to positive and negative values for ϵ and give 'upflow' and 'downflow' at some point in Ω. Stability here is in the sense of linear theory, $\sigma(\epsilon,\xi,\zeta) > 0$ means instability. Subcritical convection exists when

$$0 > \mathscr{R} - \mathscr{R}_c = \mathscr{R}_1\epsilon + \mathscr{R}_2\epsilon^2 + O(\epsilon^3). \quad (4.20)$$

Hence when ϵ is sufficiently small, $\epsilon\mathscr{R}_1 < 0$ means that the convection is subcritical and $\epsilon\mathscr{R}_1 > 0$ means that the convection is supercritical.

THEOREM 1. *Suppose that conditions A hold and that* $\xi^2+\zeta^2$ *and* $\epsilon^2 > 0$ *are sufficiently small.*

(i) *When* $\xi^2+\zeta^2 = 0$, *the 'upflow' and 'downflow' solutions are both stable* (Yudovich 1967*b*).

(ii) *Let* $\xi^2+\zeta^2 > 0$. *Then the supercritical solution is stable. There exists* $|\epsilon_*| > 0$ *such that the subcritical solution is unstable when* $|\epsilon| < |\epsilon_*|$ *and is stable when* $|\epsilon| > |\epsilon_*|$. *Moreover, as* $\xi^2+\zeta^2 \to 0$

$$\epsilon_* = -\mathscr{R}_1/(2\mathscr{R}_2 + \mathscr{R}_1\phi) + O(\epsilon_*^2) \quad (4.21)$$

and $\mathscr{R} - \mathscr{R}_c$ *is a local minimum at* $\epsilon = \epsilon_*$.

Proof of Theorem 1

Using lemma 2 we may rewrite (4.1) as

$$\sigma(\epsilon,\xi,\zeta)b^2 = -\mathscr{R}_1\epsilon - \{2\mathscr{R}_2 + \mathscr{R}_1\phi\}\epsilon^2 + O(\epsilon^3). \quad (4.22)$$

When $\xi^2+\zeta^2 = 0$, $\mathscr{R}_1 = 0$, $\mathscr{R}_2 > 0$ (Yudovich 1967*a*, Fife & Joseph 1969). Hence, $\sigma < 0$ for small ϵ. This proves (i). To prove (ii) we first note that for the supercritical solution, $\epsilon\mathscr{R}_1 > 0$ and $\sigma < 0$. The subcritical solution has $\epsilon\mathscr{R}_1 < 0$ and $\sigma > 0$ for small ϵ, but $\sigma = 0$ when

$$\epsilon^2 = \{-\epsilon\mathscr{R}_1(1+\epsilon\phi) + O(\epsilon^3)\}/2\mathscr{R}_2. \quad (4.23)$$

Since ϕ is a bounded quantity independent of ϵ, $\mathscr{R}_2 > 0$ and $\epsilon\mathscr{R}_1 < 0$, (4.23) may be solved for $\epsilon = \epsilon_*$ with $|\epsilon_*|^2 > 0$ when $\mathscr{R}_1$ is sufficiently small. In particular, $\mathscr{R}_1 \to 0$ with $\xi^2+\zeta^2$ and leads to the formula (4.21). From (4.22) and (4.23) we see that $\sigma < 0$ when $|\epsilon| > |\epsilon_*|$ and $\epsilon\mathscr{R}_1 < 0$. The minimizing property mentioned in the last sentence of the theorem follows from comparison (4.21) and (4.20).

It is worth noting that the cubic operator $\mathbf{G}[\mathbf{q},\mathbf{q},\mathbf{q}]$ first enters the perturbation problem through (4.7), (4.10) for second derivatives. Such non-linearities are characteristic for problems with temperature-dependent material coefficients (like viscosity). The inertial terms are quadratic and make no contribution to the operator **G**. Inspection of (4.22) shows that non-linearities of order higher than quadratic affect σ only through $\mathscr{R}_2$ and terms of $O(\epsilon^3)$ which do not enter our analysis. The effect of the cubic non-linearities cannot be ignored since by (4.13)

$$\mathscr{R}_2 - \mathscr{R}_2(\mathbf{G} = 0) = \langle \mathbf{q}^{\Delta} . \mathbf{G}[\mathbf{q}, \mathbf{q}, \mathbf{q}] \rangle.$$

In the case of no heat sources $\xi = 0$ and $\mathbf{q}^{\Delta} = \mathbf{q}$, we have, after integration by parts, that

$$\mathscr{R}_2 - \mathscr{R}_2(\mathbf{G} = 0) = \langle \mathbf{q} . \mathbf{G} \rangle = -(\mathscr{P}\zeta/\mathscr{R}_c)^2 \langle \theta^2 \hat{\gamma} \gamma'' \mathbf{d} : \mathbf{d} \rangle. \tag{4.24}$$

This formula shows how the second ϵ derivative of $\mathscr{R}$ is strongly influenced by the curvature of the viscosity function (γ''). Of course, (4.24) is small when ζ is small. But this is a consequence of the way in which ζ is said to appear in the viscosity law (2.2), and in general one should allow for curvature independent of slope. The effect of $\mathscr{R}_2$ is similar for the other material non-linearities, and it will be wiped out by the usual linear approximations which are used for the material non-linearities.

To find linear stability for subcritical convection at larger values of ξ^2 and ζ^2, we would need to consider $O(\epsilon^3)$. But for larger ϵ the possibility that the perturbation will carry some eigenvalues from $\mathrm{Re}\,[\sigma(0)] < -a^2$ to $\mathrm{Re}\,[\sigma(\epsilon)] > 0$ would also need to be excluded (see figure 3).

The criterion (4.21) shares much with one obtained by Busse (1967) for the layer problem set on a class of almost periodic functions. His criterion differs from the one given here in that, in agreement with earlier work for the viscosity problem (Palm 1960, Segel & Stuart 1962, Palm & Øiann 1964), he finds that the supercritical branch (A in figure 1) is unstable. This result is also said, by Krishnamurti (1968), to hold for the heat source problem.

The discrepancy between the results derived here and those which hold in the fluid layer can be traced to the degeneracy of the problem set in layers. In the intersection of almost periodic functions and functions of the plan-form type where Busse's linearized treatment starts, the eigenvalue $\mathscr{R}_c$ is not simple but instead has an infinite multiplicity.

When $\mathscr{R}_2$ is simple and $\mathscr{R}$ is nearby, one can destabilize the convection with only one disturbance mode, and this mode is very nearly proportional to the convection itself. But when $\mathscr{R}_c$ is a multiple eigenvalue, then all of the eigenfunctions belonging to $\mathscr{R}_c$ are available to destabilize the convection. This is why one can knock out supercritical hexagons (on A of figure 1) with rolls but not with hexagons (Palm 1960, Segel & Stuart 1962, Busse 1962, Palm & Øiann 1964).

Of course it is possible to *make* $\mathscr{R}_c$ a simple eigenvalue in the layer by requiring solutions to be invariant to certain translation and rotation groups (Yudovich 1966). The simplicity here, however, has considerably less force than in the true bounded domain, since there is not the guarantee that nature will submit to the symmetry restriction required of the mathematics.

The question raised is an intriguing one since, even in layers of large horizontal extent, the eigenvalue $\mathscr{R}_c$ can be simple. Yet intuition suggests that such bounded layers should not differ sensibly from infinite layers. Indeed the perturbation analyses of Newell & Whitehead (1969) and Segel (1969) attempt to show how slow spatial and time variations can be introduced into the layer so as to allow one to 'fit' the convection into a box. But the mechanics of this transition from the bounded domain to the infinite domain is imperfectly understood, and as argued in the next sections, must involve bifurcations from feeble convection rather than conduction.

5. Remarks about Bénard convection in bounded domains

It would be desirable to understand better the degree of pervasiveness of simplicity for $\mathscr{R}_c$ in the bounded domain, but I do not known how to continue the discussion at a general level.

Some things can, however, be shown for the Bénard problem ($\xi = \zeta = 0$). Its relevance to the present discussion is this: if $\mathscr{R}_0 = \mathscr{R}_c(0, 0, 0)$ is a simple eigenvalue, then perturbation theory shows that $\mathscr{R}_c(0, \xi, \zeta)$ is simple when $\xi^2+\zeta^2$ is small.

Consider the Bénard problem in a vertical cylinder Ω of (dimensionless) height 1 and cross-section $\mathscr{A}$ of arbitrary shape. The vertical side wall is designated by $\mathscr{S}$; the unit outward normal to $\mathscr{S}$ is $\mathbf{m}$, and $\mathbf{s}$ is a unit tangent vector on curves of intersection of horizontal planes and the cylinder wall $\mathscr{S}$.

It will lighten the work to restrict our attention to problems in which the vertical vorticity vanishes,

$$\mathbf{k}.\mathrm{curl}\,\mathbf{u} = \partial_x v - \partial_y u = \zeta = 0.$$

Our interest here is in the linear Bénard problem, that is problem (2.1) with $\gamma = 1$, $d\tilde{T}/dz = -1$, $(\mathbf{u}.\nabla)\mathbf{u} = (\mathbf{u}.\nabla)\theta = 0$ and $\partial/\partial t = \sigma$. Then from this specialized version of (2.1 a) one finds that $\sigma\zeta = \Delta\zeta$, and if either ζ or its normal derivative vanishes at the boundary, then $\sigma = -\langle|\nabla\zeta|^2\rangle/\langle\zeta^2\rangle$. It follows that no neutral or growing solution $\sigma \geqslant 0$ can have $\zeta \not\equiv 0$. (The possibility that $\zeta =$ constant could not be allowed unless the cylinder was in rigid rotation.)

The value of ζ on a rigid top or bottom of the cylinder is zero. On a free surface top or bottom, the normal derivative of ζ vanishes. On the side wall $\mathscr{S}$,

$$\mathbf{u} = \mathbf{m}u_m + \mathbf{s}u_s + \mathbf{k}w, \quad u_m = 0|_{\mathscr{S}}. \tag{5.1}$$

Noting that $\partial u_m/\partial s = 0|_{\mathscr{S}}$, one finds that $\mathbf{k}.\mathrm{curl}\,(\mathbf{k}w) = \mathbf{k}.\mathrm{curl}\,(\mathbf{m}u_m) = 0$. Then using the relation $\partial\mathbf{s}/\partial s = -\mathbf{m}/\rho$ where $1/\rho$ is the curvature, we find that

$$\zeta = \frac{\partial u_s}{\partial m} + \frac{u_s}{\rho}\bigg|_{\mathscr{S}}. \tag{5.2}$$

If $\zeta = 0|_{\mathscr{S}}$ or $\partial\zeta/\partial m = 0|_{\mathscr{S}}$ and the top and bottom are rigid or free surfaces, then every growing and neutral solution has $\zeta \equiv 0$ in the cylinder.

If $\zeta \equiv 0|_{\Omega}$ in the cylinder, the velocity field is purely poloidal, and there exists a 'potential' χ and an operator $\boldsymbol{\delta}$ such that

$$\mathbf{u} = \boldsymbol{\delta}\chi = \nabla\partial\chi/\partial z - \mathbf{k}\Delta\chi. \tag{5.3}$$

Most velocity fields in bounded domains will not allow fields with vanishing vertical vorticity. For example, one cannot ordinarily require four conditions on the three components of velocity, namely that all three velocity components vanish on $\mathscr{S}$ and, at the same time, require that $0 = \partial u_s/\partial m$. For the general problem one must introduce a toroidal field ($\mathbf{u} = \mathbf{k} \times \operatorname{grad} \Psi$), and the resulting problem (for χ and Ψ) is not separable.

A three-dimensional poloidal field can be found, however, in the free surface box. On this box the normal component of velocity and also (since the tangential stress is zero) the normal derivative of the tangential velocities vanish. One can satisfy (5.2) in the box because $\rho = \infty$, and the condition $\partial u_s/\partial m = 0$ is not 'extra'.

Suppose $\zeta \equiv 0|_\Omega$. Then
$$\Delta^2\chi - \mathscr{R}\theta = 0, \tag{5.4a}$$
$$\Delta\theta - \mathscr{R}\Delta_2\chi = 0, \tag{5.4b}$$

where $\Delta_2 = \partial^2_{xx} + \partial^2_{yy}$. These equations come from the pair for w and θ which express (2.1b) and $\mathbf{k}.\text{curl}^2$ (2.1a) when $\partial_t = 0$, $\gamma = 1$, using $w = -\Delta_2\chi$. On the free top and bottom of the box, we have

$$\chi = \partial^2\chi/\partial z^2 = 0|_{z=0,1}. \tag{5.4c}$$

On the free side wall with its normal in the direction x, the conditions $u = 0$ and $\partial w/\partial x = 0$ are equivalent to

$$0 = \partial\chi/\partial x = \partial^3\chi/\partial x^3|_{x=0,l_x}, \tag{5.4d}$$

and, similarly, in the direction y,

$$0 = \partial\chi/\partial y = \partial^3\chi/\partial y^3|_{y=0,l_y}. \tag{5.4e}$$

Suppose also that
$$\theta = 0|_{z=0,1} \quad \text{and} \quad \partial\theta/\partial m = 0|_{\mathscr{S}}. \tag{5.4f, g}$$

Then the problem can be solved by solutions of the form

$$\begin{bmatrix}\chi\\ \theta\end{bmatrix} = \sum_{n=1}\sum_{\nu=1}\begin{bmatrix}\chi_{\nu n}\\ \theta_{\nu n}\end{bmatrix} f_\nu(x,y)\sin n\pi z, \tag{5.5}$$

where
$$\Delta_2 f_\nu + \lambda_\nu^2 f_\nu = 0, \quad \partial f/\partial m = 0|_{\mathscr{S}}. \tag{5.6}$$

Here, $\mathscr{S}$ is the rectangle of sides (l_x, l_y) and

$$f_\nu = \cos\frac{\mu\pi x}{l_x}\cos\frac{\alpha\pi y}{l_y}, \quad \lambda_\nu^2 = \pi^2\left(\frac{\mu^2}{l_x^2}+\frac{\alpha^2}{l_y^2}\right), \tag{5.7a}$$

where α and μ are integers and the λ_ν are ordered by size ($\lambda_1 \leqslant \lambda_2 \ldots$). One finds by introducing (5.5) into (5.4a, b) that

$$\mathscr{R}^2\lambda_\nu^2 = (n^2\pi^2 + \lambda_\nu^2)^3. \tag{5.7b}$$

These are the required eigenvalues.

$\mathscr{R}^2$ is a simple eigenvalue of (5.4) provided λ_ν^2 is a simple eigenvalue of (5.6) and provided that for the pair (n, ν) there is not a second pair $\tilde{n}$, $\tilde{\nu}$ such that $\mathscr{R}^2_{n\nu} = \mathscr{R}^2_{\tilde{n}\tilde{\nu}}$. Equation (5.7) shows that only one f_ν belongs to λ_ν^2 if l_x^2/l_y^2 is irrational. But one can always find rational l_x^2/l_y^2 to satisfy the relation

$$\frac{\mu^2-\tilde{\mu}^2}{\tilde{\alpha}^2-\alpha^2} = \frac{l_x^2}{l_y^2},$$

when the left side is given. These values make λ_ν^2 a multiple eigenvalue.

The smallest of the eigenvalues $\mathscr{R}_0^2$ is associated with $n^2 = 1$ and is given by

$$\mathscr{R}_0^2 = \min_{\nu=1,2,\ldots} \frac{(\pi^2+\lambda_\nu^2)^3}{\lambda_\nu^2} = \frac{\pi^4(1+\hat{\mu}^2/l_x^2+\hat{\alpha}^2/l_y^2)^3}{\hat{\mu}^2/l_x^2+\hat{\alpha}^2/l_y^2}.$$

This smallest eigenvalue is simple in most domains. The distance between this smallest value and the next largest value

$$\min_{n,\alpha,\mu}\left\{\frac{(n^2+\mu^2/l_x^2+\alpha^2/l_y^2)^3}{\mu^2/l_x^2+\alpha^2/l_y^2} - \frac{(1+\hat{\mu}^2/l_x^2+\hat{\alpha}^2/l_y^2)^3}{\hat{\mu}^2/l_x^2+\hat{\alpha}^2/l_y^2}\right\} \tag{5.8}$$

tends to zero as l_x^2 and l_y^2 become large (both the second and first term of (5.8) tend to 27/4). On the other hand, when $l_x^2+l_y^2$ is small, the distance between successive eigenvalues is very large.

In the free surface box the principal eigenfunction can be a roll with its axis along $x(\mu = 0)$ or along $y(\alpha = 0)$, or it can be a rectangle ($\mu \neq 0 \neq \alpha$). For example, with $\epsilon \to 0$ $l_y^2/l_x^2 = 2+\epsilon$, $l_y^2 = 34$, $\lambda_\nu = \mu^2(2+\epsilon)/34+\alpha^2/34$, the principal eigenvalue is

$$\mathscr{R}_0^2 = \pi^4\left(\frac{3}{2}+\frac{2\epsilon}{17}\right)^3 \Big/ \left(\frac{1}{2}+\frac{2\epsilon}{17}\right),$$

which corresponds to $\alpha = 2$, $\mu = 3$. Unlike the rigid box (Davis 1967), the principal eigenfunction in the free box does not show the same strong tendancy to align itself as a roll parallel to the short axis.

The free surface cylinder which was studied by Liang, Vidal & Acrivos (1969) differs from the box in one important respect: it is not possible to have non-axisymmetric solutions with vanishing vertical vorticity.

Consider the round cylinder with a free top and bottom and let $\theta = 0|_{z=0,1}$. Then, by using (5.4c) and (5.4a, b) a bootstrap argument leads to the conclusion that all solutions of this problem have a z dependence of the form (5.5). Thus $\chi_\nu(r,\theta)$ and $\theta_\nu(r,\phi)$ must satisfy

$$(\Delta_2 - \nu^2\pi^2)^2\chi_\nu - \mathscr{R}\theta_\nu = 0, \tag{5.9a}$$

$$(\Delta_2 - \nu^2\pi^2)\theta_\nu - \mathscr{R}\Delta_2\chi_\nu = 0, \tag{5.9b}$$

where

$$\Delta_2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2},$$

and (r,ϕ,z) are cylindrical co-ordinates.

When the side wall is free, the radial velocity and the two strain rates d_{rz} and $d_{r\phi}$ must all vanish at $r = r_0$. But in addition we have already assumed that the vertical vorticity (5.2) must vanish at r_0, and this is compatible with $d_{r\phi} = 0|_{r_0}$ only if $u_\phi = 0|_{r_0}$. It follows that three independent conditions hold at $r = r_0$,

$$\partial\chi_\nu/\partial\phi = \partial\chi_\nu/\partial r = \partial(\Delta_2\chi_\nu)/\partial r = 0|_{r_0}, \tag{5.10c, d, e}$$

and three independent conditions plus a condition on θ_ν at r_0 overdetermine problem (5.9). Here, as in the rigid side wall problem mentioned earlier, one needs a toroidal as well as a poloidal field.

On the other hand, if one's attention is restricted to axisymmetric solutions, the condition $u_\phi = 0$ is an identity (since $\chi_\nu = \chi_\nu(r)$), and it is not necessary to require that $\partial\chi_\nu/\partial\phi = 0|_{r_0}$. Then if $\partial\theta/\partial r = 0|_{r_0}$, one can satisfy the sixth-order problem arising from (5.9*a*, *b*) and (5.10*d*, *e*) with

$$\Delta_2\chi_\nu = -\lambda_\nu^2\chi_\nu, \quad \partial\chi_\nu/\partial r = 0|_{r_0}, \tag{5.11}$$

that is, with $\chi_\nu = J_0(\lambda_\nu r)$, where the λ_ν are the positive roots of $d(J_0(\lambda_\nu r))/dr = 0|_{r_0}$, and

$$\mathscr{R}^2\lambda_\nu^2 = (\lambda_\nu^2+\nu^2\pi^2)^3. \tag{5.12}$$

The λ_ν are simple eigenvalues of (5.11) and the principal eigenvalue $\mathscr{R}^2$ of (5.12) is also simple. This is the linear part of the problem which was treated by Liang *et al.* (1969). There is as yet no published demonstration that the axisymmetric mode of convection gives the smallest value for $\mathscr{R}^2$.

Explicit Bessel function solutions of the *rigid* side wall problem are easily obtained from (5.9) when the motion is axisymmetric. On the rigid side wall $w = u_r = 0$ leads one to the conditions

$$\Delta_2\chi_\nu = d\chi_\nu/dr = 0|_{r_0}. \tag{5.13a, b}$$

Suppose that

$$\partial\theta_\nu/\partial r + h\theta_\nu = 0|_{r_0}, \tag{5.14}$$

where h is a positive constant. Here $h = 0$ gives the insulated side wall and $h = \infty$, the conducting side wall. Using (5.9*a*), we convert (5.14) into a condition on χ.

$$\frac{\partial}{\partial r}(\Delta_2-\nu^2\pi^2)^2\chi_\nu + h(\Delta^2-\nu^2\pi^2)^2\chi_\nu = 0|_{r_0}. \tag{5.13c}$$

The sixth-order equation

$$(\Delta_2-\nu^2\pi^2)^3\chi_\nu - \mathscr{R}^2\Delta_2\chi_\nu = 0 \tag{5.15}$$

is reduced to (5.12) by the Bessel functions $J_0(\lambda_\nu r)$. Equation (5.12) is a cubic polynomial with real coefficients. It has three roots and the complex roots occur in conjugate pairs. Let $\lambda_{\nu 1}^2$, $\lambda_{\nu 2}^2$ and $\lambda_{\nu 3}^2$ be the three roots. The linear combination

$$\chi_\nu = A_1 J_0(\lambda_{\nu 1} r) + A_2 J_0(\lambda_{\nu 2} r) + A_3 J_0(\lambda_{\nu 3} r)$$

is the general solution of (5.15). It solves the problem, that is (5.13*a*, *b*, *c*) and (5.15), for the special values (eigenvalues) which appear as roots of the equation $\Delta(\nu, h, \mathscr{R}) = 0$ where Δ is the determinant whose columns are

$$\lambda_{\nu i}^2 J_0(\lambda_{\nu i} r_0),$$

$$\frac{d}{dr} J_0(\lambda_{\nu i} r_0),$$

$$(\lambda_{\nu i}^2+\nu^2\pi^2)^2\left\{\frac{d}{dr}J_0(\lambda_{\nu i} r_0) + hJ_0(\lambda_{\nu i} r_0)\right\}$$

and $i = 1, 2, 3$. In figure 4 we have plotted some of the values $\mathscr{R}(h, r_0)$. It is striking how sensitive these values are to change in r_0, even in disks when diameters are twice their height. In all cases the smallest $\mathscr{R}$ has $\nu = 1$.

One can, in the same way but with six Bessel functions rather than three, resolve the problem of axisymmetric convection in an annular ring bounded by rigid side walls but free at the top and bottom. The experimental work of Goldstein & Graham (1969) shows that such configurations can be achieved in the laboratory.

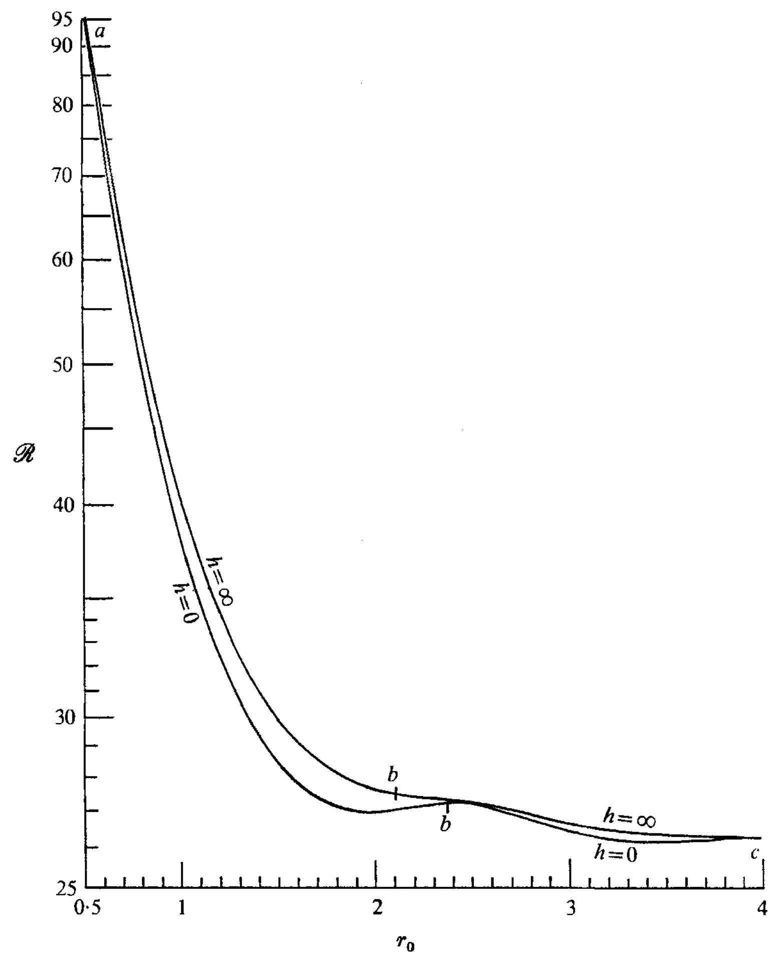

FIGURE 4. The critical Rayleigh number ($\mathscr{R}_c^2 = \alpha g \Delta T l^3/\nu_0 \kappa$) for the instability of the constant gradient conduction solution to axisymmetric convection in a disk with a free top and bottom and a rigid side wall. The value $h = 0$ is the insulating side wall. The value $h = \infty$ is the conducting side wall. The arcs *ab* have one cell, *bc* has two cells. The co-ordinate r_0 gives the ratio of the radius of the disk to its height l.

6. Concluding remarks

Feeble convection in most bounded domains is dominated by the consequences of the simplicity of $\mathscr{R}_c$. Two $(\pm\epsilon)$ steady convective solutions branch off conduction at $\mathscr{R} = \mathscr{R}_c$ (see figure 1). One of these branches can have a subcritical arc with $\mathscr{R} < \mathscr{R}_c$. Both branches of convection can be stable if $\mathscr{R}_c$ is simple. But if $\xi^2 + \zeta^2$ is small, the subcritical branch is stable if $|\epsilon| > |\epsilon_*| > 0$. Hence in bounded domains it is possible to realize both the 'up' and 'down' solutions when $\mathscr{R} > \mathscr{R}_c$ (see Liang *et al.* (1969) for the analysis and experiment for convection in a round cylinder), but only one of these solutions can be realized when $\mathscr{R} < \mathscr{R}_c$. Only this subcritical motion has a uniquely determined sign.

In fluid layers the supercritical branch of convection is unstable, and the mechanics of the transition from stability for the supercritical branch in the bounded domain to instability in the layer needs to be explained.† My ideas about this transition are speculative. Let us think of cylinder Ω of cross-section $\mathscr{A}$ and height one. Let d be the minimum diameter of $\mathscr{A}$. One can have cylinders Ω of large horizontal extent (d) which retain the property of simplicity for $\mathscr{R}_0 = \mathscr{R}_c(\zeta = 0, \xi = 0)$. (The reader can construct such an example for himself in the free surface box, cf. (5.8).) In these layer-like domains, the supercritical branch of convection is stable when $|\epsilon| < \delta(d)$. But it is likely that $\delta(d) \to 0$ with $1/d$. The reason that this is likely has to do with spectral crowding.

By spectral crowding I mean nothing deeper than the elementary observation that as d is increased, the distance between successive eigenvalues decreases. This process is exhibited explicitly for the free surface box by (5.8), which gives the distance between the principal eigenvalue $\mathscr{R}_0$ of (5.4) and its nearest neighbour. This distance tends to zero as l_x^2 and l_y^2 tend to infinity. Into any fixed interval $L > |\mathscr{R} - \mathscr{R}_0|$, we can crowd $n \geqslant N(l_x, l_y)$ eigenvalues of (5.4) and $N \to \infty$ as l_x, $l_y \to \infty$.

A similar crowding of eigenvalues can be demonstrated and is relevant when considering the spectral problem for the stability of conduction. Then in the 'free surface' box the 'growth rates' $\sigma(\mathscr{R})$ are determined as eigenvalues of the problem

$$\sigma\Delta\chi = -\mathscr{R}\theta + \Delta^2\chi, \tag{6.1a}$$

$$\sigma\mathscr{P}\theta = -\mathscr{R}\Delta_2\chi + \Delta\theta, \tag{6.1b}$$

and the boundary conditions (5.4). These appear as roots of the equation ($n \geqslant 1$, $\lambda_\nu^2 > 0$)

$$\sigma\mathscr{P} = \frac{\mathscr{R}^2\lambda_\nu^2 - (\sigma + n^2\pi^2 + \lambda_\nu^2)(n^2\pi^2 + \lambda_\nu^2)^2}{(n^2\pi^2 + \lambda_\nu^2)(\sigma + n^2\pi^2 + \lambda_\nu^2)},$$

where λ_ν^2 is given by (5.7). To see the nature of the eigenvalues as the horizontal extent of the box is made large, we put

$$\tilde{\sigma} = \tilde{\sigma}_{n,\mu,\alpha}(l_x, l_y, \mathscr{R}^2) = \sigma\mathscr{P}, \quad \mathscr{P} \to \infty \quad \text{(for convenience)}.$$

Then noting that by the argument leading to (5.8), we have

$$\mathscr{R}_0^2 \simeq \tfrac{27}{4}\pi^4,$$

† In physical problems which are set on bounded domains, it is sometimes more natural to think of a region of transition rather than of a well-defined boundary. Even when the boundary is well defined, it can happen that the boundary conditions are not of a nice kind or very easily posed. In this regard, the inlet and outlet of a pipe or a finite fluid layer without confining side walls come to mind. Such problems are frequently best treated by extending them as mathematical problems onto infinite domains. The problem of extension then mirrors the chaotic conditions which can prevail at the 'edges' of the physical system. The mathematical difficulties which can arise in the extension, like loss of simplicity, discreteness of the spectrum and compactness of operators, are not consequences of bad mathematics but stem from the complexity of physics. The analysis of the extended problem could easily have more relevance than an analysis of a bounded domain problem with artificially 'nice' conditions set on edges. This may be the case, for example, in experiments like Graham's (1933), in which the edges of the gas layer were not confined.

and find that when $\mathscr{R} = \mathscr{R}_0$,

$$\tilde{\sigma} = \frac{\mathscr{R}_0^2(\mu^2/l_x^2 + \alpha^2/l_y^2)}{\pi^2(n^2 + \mu^2/l_x^2 + \alpha^2/l_y^2)^2} - \pi^2(n^2 + \mu^2/l_x^2 + \alpha^2/l_y^2),$$

where n, α and μ are integers and n and $\alpha^2 + \mu^2$ are not zero. Zero is the largest value which is possible for $\tilde{\sigma}$, and it is taken on for the value $n = 1$, $\alpha = \hat{\alpha}$ and $\mu = \hat{\mu}$. All the other eigenvalues $\tilde{\sigma}_{n\mu\alpha}(l_x, l_y, \mathscr{R}_0)$ are negative, the distance between $\tilde{\sigma} = 0$ and the next smallest eigenvalue tends to zero as l_x and l_y are made large.

Now in figure 3 we draw a circle of radius L around the origin. No matter how small this circle, it is possible by enlarging the box to inject into this circle an arbitrarily large number of eigenvalues $\mathscr{R}$ of (5.4) or $\sigma(\mathscr{R}_0)$ of (6.1). It follows that the value a^2 shown in figure 3 must tend to zero as the domain is enlarged. Moreover, the nearest neighbours of the largest eigenvalue $\sigma = \sigma_{i\hat{\alpha}\hat{\mu}}(\mathscr{R}_0) = 0$ can now change sign in the ϵ perturbation. In this way eigenvalues $\sigma(\epsilon)$ which were negative when $\epsilon = 0$ can become positive for $|\epsilon| > 0$ and lead to instability of the supercritical convection.

A similar remark holds for perturbations of the spectrum which arise from the inevitable fluctuations which go with physical systems. If the eigenvalues of the ideal mathematical system are sufficiently dense near their principal values, such fluctuations will cause them to coalesce and change their order. In this sense the crowding of the spectrum, even when $\mathscr{R}_c$ is 'ideally' simple, can be an effective 'loss of simplicity', and for it the infinite domain analysis may be most appropriate.

When the spectrum is separated, however, and $\mathscr{R}_c$ is simple and ϵ is small the convection is just a small perturbation of the eigenfunction belonging to $\mathscr{R}_c$. Then if $\mathscr{R}$ is very slowly changed, the only way in which new steady solutions could become available is by branching off feeble convection rather than off conduction. Viewed in this way, the appearance of hexagons in a cylinder (under steady exterior constraints) could only arise from bifurcation of *convection.* As $\epsilon \to 0$ and $\mathscr{R}(\epsilon) \to \mathscr{R}_c$ only one form of infinitesimal convection would be available, and it is not hexagonal.

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